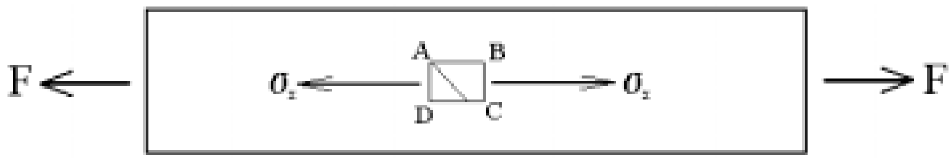
APPENDIX – Derivation of the Stress Transformation Equations

Consider the bar shown in Figure 1 subjected to an axial force F through the centroid of the cross section. A rectangular element ABCD within the bar will be subjected to direct stresses σ_x = F/(AD)t on faces AD and BC of the element.



thickness t

Figure 1

Consider now the stresses on plane AN, inclined at angle ϕ to AD. For equilibrium of the triangular element to be maintained, a direct stress σ_ϕ and a shear stress τ_ϕ will be required to act on the plane as shown in Figure 2.

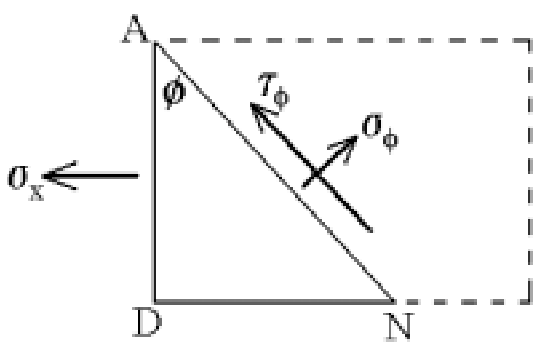


Figure 2

Consider the equilibrium of the triangular element AND.

Resolving forces in the direction of σ_ϕ:

$$\sigma_\phi ANt = \sigma_x ADt \cos\phi$$

$$\sigma_\phi = \sigma_x \frac{AD}{AN} \cos\phi$$

$$\sigma_\phi = \sigma_x \cos^2\phi$$

Resolving forces in the direction of τ_ϕ:

$$\tau_\phi ANt = -\sigma_x ADt \sin\phi$$

$$\tau_\phi = -\sigma_x \frac{AD}{AN} \sin\phi$$

$$\tau_\phi = -\sigma_x \cos\phi \sin\phi$$

$$\tau_\phi = -\frac{1}{2}\sigma_x \sin 2\phi$$

Inspection of the results shows that:

when $\phi = 0^o$,	$\sigma_\phi = \sigma_x$ (its maximum value)
when $\phi = 90^o$,	$\sigma_\phi = 0$
when $\phi = 45^o$ or 135^o,	$\sigma_\phi = \sigma_x / 2$
when $\phi = 0$ or 90^o	$\tau_\phi = 0$
when $\sin 2\phi = +/-1$	τ_ϕ is a minimum
i.e. when $2\phi = 90^o$ or 270^o ;	$\phi = 45^o$ or 135^o,
$\tau_{45} = -\sigma_x / 2$	$\tau_{135} = +\sigma_x / 2$

The analysis shows that for materials whose shear strength is less than half the tensile strength, direct tensile loading results in failure along the planes of maximum shear stress.

<u>The general two dimensional stress state</u>

In general an element within a two dimensional stress field may be subjected to direct and shear stresses on all its faces. Also if the element is small, defining behaviour at a point in the plane, then the stresses on parallel faces must be equal and opposite for equilibrium. This gives the general state of stress at a point shown in Figure 3.

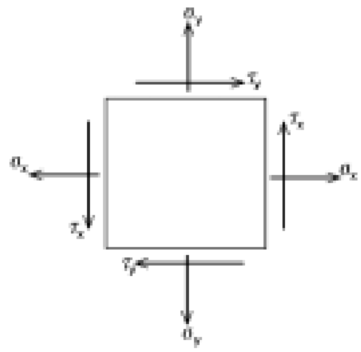


Figure 3

Complementary Shear Stresses

In order that the element shown in Figure 3 is in rotational equilibrium we can deduce that the sense of the shear stresses on adjacent faces must be opposite to each other.

$$\tau_x = \tau_y \qquad (1)$$

This condition, which must be satisfied by any pair of adjacent shear stresses, is important and is called 'the condition of complementary shear stress'.

Consider the general state of stress acting on the element ABCD of thickness 't' (Figure 4). We wish to find the direct and shear stresses on some plane AN inclined at angle o to AD.

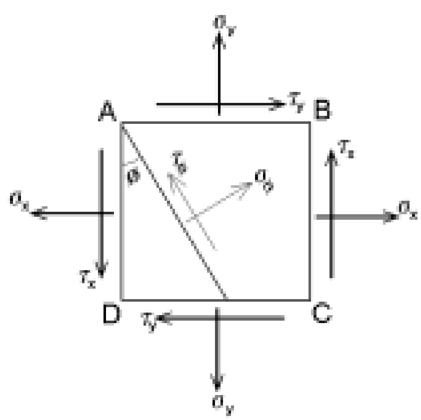


Figure 4

The free body diagram for the wedge AND is shown in Figure 5. Note that forces, not stresses, are used in the free body diagram. The stresses are converted to forces by multiplying them by the areas over which they are distributed.

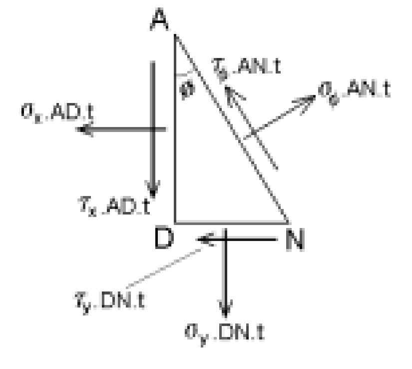


Figure 5

Resolving in the direction of σ_ϕ:

$$\sigma_\phi ANt = \sigma_x ADt \cos\phi + \tau_x ADt \sin\phi + \sigma_y DNt \sin\phi + \tau_y DNt \cos\phi$$

Noting that:

$$\frac{AD}{AN} = \cos\phi$$

$$\frac{DN}{AN} = \sin\phi$$

$$\tau_y = \tau_x$$

Then:

$$\sigma_\phi = \sigma_x \cos^2\phi + \sigma_y \sin^2\phi + 2\tau_x \sin\phi\cos\phi$$

Therefore:

$$\sigma_\phi = \frac{1}{2}\sigma_x(1+\cos 2\phi) + \frac{1}{2}\sigma_y(1-\cos 2\phi) + \tau_x \sin 2\phi$$

And:

$$\sigma_\phi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\phi + \tau_x \sin 2\phi \qquad (2)$$

Resolving in the direction of τ_ϕ:

$$\tau_\phi ANt = -\sigma_x ADt \sin\phi + \tau_x ADt \cos\phi + \sigma_y DNt \cos\phi - \tau_y DNt \sin\phi$$

Thus:

$$\tau_\phi = -\sigma_x \cos\phi \sin\phi + \tau_x \cos^2\phi + \sigma_y \cos\phi \sin\phi - \tau_y \sin^2\phi$$

And:

$$\tau_\phi = -\frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\phi + \tau_x \cos 2\phi \qquad (3)$$

Equations (2) and (3) are known as the **STRESS TRANSFORMATION EQUATIONS**.