

Stress - strain relationships

In most structural engineering design problems it is important to determine both the strength and stiffness of the design. Usually this requires a knowledge of the material behaviour which is determined experimentally. Many materials behave approximately linearly over the initial part of the stress - strain graph. It is further observed that the linear region often coincides with the elastic region (in which the material returns to its original size when unloaded).

The term LINEAR - ELASTIC BEHAVIOUR is used in this context and most engineering design is based on this assumption.

Uniaxial Linear Elasticity

If tests are carried out on a prismatic bar subjected to uniaxial tension, then the constant of proportionality between stress and strain may be found. This material property is known as the MODULUS OF ELASTICITY or ELASTIC MODULUS or YOUNG'S MODULUS (E) and the linear relationship becomes:

$$\epsilon_x = \frac{\sigma_x}{E} \quad (4)$$

This is known as HOOKE'S LAW.

In addition to the axial strain, a transverse strain is observed (tensile specimens with circular cross-section suffer a reduction in diameter). This relationship is also found to be linear and a constant of proportionality called POISSON'S RATIO (ν) is defined giving for uniaxial loading:

$$\epsilon_y = -\nu \frac{\sigma_x}{E} \quad (5)$$

Biaxial Linear Elasticity

The linear behaviour associated with uniaxial loading may be used to predict the biaxial behaviour of isotropic materials by using superposition.

Consider a rectangular element subjected to σ_x and σ_y as shown in Figure 6.

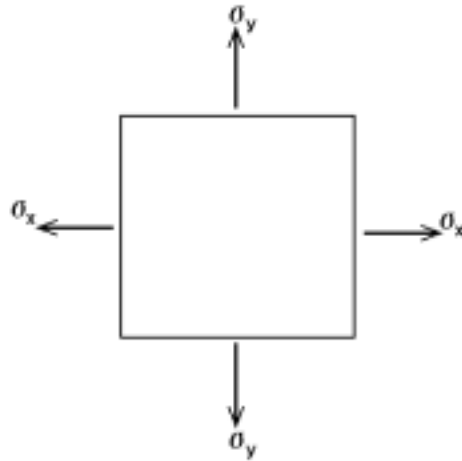


Figure 6.

Then, due to σ_x :

$$\varepsilon_x = \frac{\sigma_x}{E}$$

And:

$$\varepsilon_y = -\nu \frac{\sigma_x}{E}$$

And due to σ_y :

$$\varepsilon_x = -\nu \frac{\sigma_y}{E}$$

And:

$$\varepsilon_y = \frac{\sigma_y}{E}$$

Giving by superposition:

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad (6a)$$

(6b)

We also see that the above equations apply to stress and strain in any orthogonal set of axes and hence may also be applied in the principal directions:

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \quad (7a)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \quad (7b)$$

Further the two equations may be solved simultaneously to give the explicit form for the stresses.

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \quad (8a)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \quad (8b)$$

Shear Relationships

The linear behaviour discussed in the previous paragraphs is also found to apply in shear where the constant of proportionality between shear stress and shear strain is known as the MODULUS OF RIGIDITY or SHEAR MODULUS (G).

$$\gamma = \frac{\tau}{G} \quad (9)$$

Shear stress is necessarily biaxial (any shear stress τ_x must have a complementary shear stress τ_y). It is also found that for small deformation the relationship between shear stress and shear strain is independent of direct stress.

Three Dimensional Behaviour

The biaxial stress - strain relationships may be extended to three dimensions to give for isotropic materials:

Hooke's Law for Three Dimensional Behaviour

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu\sigma_y - \nu\sigma_z) \quad (10a)$$

$$\varepsilon_y = \frac{1}{E} (\sigma_y - \nu\sigma_x - \nu\sigma_z) \quad (10b)$$

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \nu\sigma_x - \nu\sigma_y) \quad (10c)$$

Example 2

A strain gauge measures the strain at a point on the surface of a component in a direction where the stress is 100 MPa and the transverse stress is 20 MPa. Find the strain measured if Young's Modulus is 200 GPa and Poisson's Ratio is 0.28.

$$\varepsilon = \frac{1}{E} (\sigma - \nu\sigma_t)$$

$$\varepsilon = \frac{1}{200 \times 10^9} (100 - 0.28 \times 20) \times 10^{-6}$$

$$\varepsilon = 472 \times 10^{-6} = 472 \text{ microstrain}$$

Example 3

Find the principal stresses at the rosette in Example 1 if Young's Modulus is 73 GPa and Poisson's Ratio is 0.32.

From equation (8):

$$\sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2)$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1)$$

$$\sigma_1 = \frac{73 \times 10^9}{1-0.32^2}(779 + 0.32 \times 21) \times 10^{-6}$$

$$\sigma_1 = 64 \times 10^6 \text{ N/m}^2$$

$$\sigma_1 = 64 \text{ MN/m}^2$$

$$\sigma_2 = \frac{73 \times 10^9}{1-0.32^2}(21 + 0.32 \times 779) \times 10^{-6}$$

$$\sigma_2 = 22 \text{ MN/m}^2$$

Example 4

A cylindrical component is strain gauged to measure the radial strain and the circumferential strain. The values obtained were -50 microstrain and 250 microstrain respectively at a point where the radius is 120mm. If Young's Modulus is 200 GPa and Poisson's Ratio is 0.28 determine:

- radial and circumferential stresses;
- change of radius at the point due to the strains.

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta)$$

$$\sigma_r = \frac{200 \times 10^9}{1 - 0.28^2} (-50 + 0.28 \times 250) \times 10^{-6}$$

$$\sigma_r = 4.34 \times 10^6 \text{ N / m}^2$$

$$\sigma_r = 4.34 \text{ MN / m}^2$$

$$\sigma_\theta = \frac{200 \times 10^9}{1 - 0.28^2} (250 + 0.28 \times (-50)) \times 10^{-6}$$

$$\sigma_\theta = 51.2 \text{ MN / m}^2$$

Change in radius:

$$\varepsilon_\theta = \frac{\Delta \text{circumference}}{\text{circumference}} = \frac{2\pi\Delta R}{2\pi R}$$

Therefore:

$$\Delta R = R\varepsilon_\theta$$

$$\Delta R = 120 \times 250 \times 10^{-6}$$