MECH226

VIBRATION

Section 1A – Free Vibration

Useful texts:

Den Hartog, J.P., Mechanical vibrations, 1985 ISBN 0-486-64785-4 (620.3 DEN)

Hibbeler, R.C., Engineering Mechanics Dynamics, published by Prentice-Hall, 7th edition, 1997 ISBN 0-13-741018-2 (or later editions) (620.1054 HIB)

MacCallion, H., Vibration of linear mechanical systems, published by the Longman Group, 1973 (620.3 MAC)

Meriam, J.L. and Kraige, L.G., Engineering Mechanics Vol. 2, Dynamics, 2nd Edition 1987 ISBN 0-471-84912-X (620.104 MER) (or later editions)

T235:Engineering mechanics: solids Block 8 Vibration, published by the Open University, 1990 ISBN 0-7492-6036-X (620.105 OPE)

Thomson, W.T. and Dahleh, M.D., Theory of vibration with applications, 5th edition, published by Prentice-Hall, 1998 ISBN 0-13-651068-X , 620.3

Tongue, B.H., Principles of Vibration, 1996 ISBN 0-19-510661 (620.3 TON)

1. Introduction

In order to vibrate, a system must have two characteristics: elasticity and mass. Since all solids possess both, all solids, under the right conditions, are capable of vibration. By this I mean they can experience some form of cyclic deformation or displacement.

Vibration can be useful. Without it there would be no music or clocks; many machines rely on vibration for conveying or sorting materials. However, it can also cause problems for the mechanical engineer, and much effort may be spent in reducing unwanted vibration.

Unwanted vibration causes two main problems: fatigue failure and failure due to excessive deformation.

Fatigue failure occurs when a component is subjected to sustained cyclic loads. The amplitude and mean stress may be smaller than the tensile strength of the material, but the repeated stress cycles cause crack growth, until a critical crack length is reached and the material fails. Aluminium alloys are a class of material that are particularly susceptible to fatigue failure. So even small amplitude vibration transmitted to aluminium components may eventually cause them to fail.

Excessive deformation may be caused if vibration occurs at a resonant frequency. One example is that of a washing machine with an unbalanced load. At a particular speed of rotation of the drum during the spin cycle, vibration of a large amplitude may occur, which may be sufficient to damage the machine or its supports. A famous failure caused by resonance was that of the Tacoma Narrows Bridge.

We will look more closely at the phenomenon of resonance in due course.

In general, vibration can be classed as either free (or natural) or forced. Free vibration occurs when a system is given a small displacement or deformation and then released. An example is that of a diving board; when the diver has jumped off it, the board will be set vibrating. Another example is that of a guitar string that is plucked once; it continues to vibrate when released. Forced vibration is produced when a cyclic force is applied to the system.

Both these classes of vibration can be considered to be either undamped or damped. Damping in a system causes energy to be dissipated and may arise from a number of different sources, including internal friction, within the material itself, and external friction, such as that between the system and its supports.

Vibrating systems may be classified by the number of degrees of freedom they possess. This is the number of parameters required to describe the motion of elements within the system. In this course we will consider only single degree-of-freedom systems. We will start by looking more closely at each of the four classes of vibration:

- Free vibration
- Free damped vibration
- Forced vibration
- Forced damped vibration

Later, we will look at vibrations caused by rotary imbalance, and at ways of achieving vibration isolation.

2. Free vibration of a single degree-of-freedom system

As I said above, in order to vibrate a system needs elasticity and mass. The simplest such system consists of a weight attached to a spring (Figure 1.1):

Figure 1.1 Simple spring-mass system

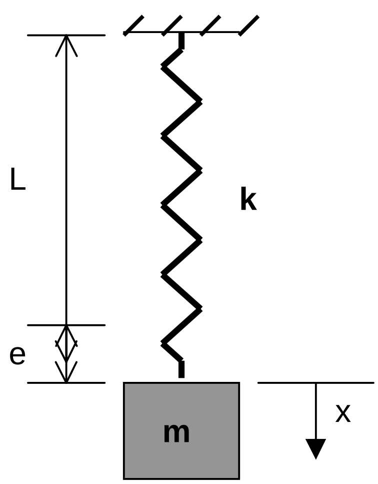
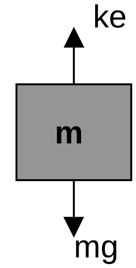


Figure 1.2. FBD of the weight



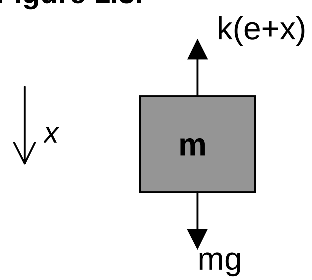
Suppose that the mass of the weight is *m*. Let the spring have a spring constant (stiffness), *k* and an unstretched length *L*. When the weight is added to the spring it stretches by an amount *e*. Then the forces acting on the weight are as shown in Figure 1.2.

If the weight is in equilibrium: $\Sigma F_x = mg - ke = 0$

$$\therefore \; ke = mg \qquad (1.1)$$

Suppose that the weight has a displacement, *x*, from the equilibrium position. The free-body diagram of the weight will be as shown in Figure 1.3. If released, the weight will move, since the system is no longer in equilibrium.

Figure 1.3.



From Newton's second law,

$\Sigma F_x = mg - k(e + x) = ma$

where *a* is the acceleration of the weight in the *x*-direction.

Using the "dot" notation, and substituting for *ke* from (1.1) gives:

$$mg - mg - kx = m\ddot{x}$$

$$\therefore m\ddot{x} + kx = 0 \qquad (1.2)$$

Equation (1.2) is a standard second order differential equation, which has the solution:

$$x = A\sin(\omega t + \varepsilon) \qquad (1.3)$$

where A and ε are constants which depend on the initial conditions, and $\omega^2 = \frac{k}{m}$

(1.4)

(Check that it *is* a solution by differentiating equation (1.3) twice to find $\ddot{x}$)

For example, suppose that the mass is given an initial displacement X_0 from the equilibrium position and then released from rest. The initial conditions are that at $t = 0$, $x = X_0$, and the velocity $\dot{x} = 0$.

Putting in these conditions:

At $t = 0$, $x = A\sin(\omega t + \varepsilon) = A\sin\varepsilon = X_0$ (i)

$\dot{x} = A\omega\cos(\omega t + \varepsilon) = A\omega\cos\varepsilon = 0$ (ii)

Since neither A nor ω can be 0 (A = 0 implies no motion at all, and ω = 0 means that there is no vibration), condition (ii) gives $\cos\varepsilon = 0$ or $\varepsilon = \pi/2$.

Putting $\varepsilon = \pi/2$ into condition (i) gives $A = X_0$

So the displacement can be written: $x = X_0 \sin(\omega t + \frac{\pi}{2}) = X_0 \cos\omega t$

which, plotted as a function of time, gives Figure 1.4. This is a typical displacement-time curve for an object which is vibrating.

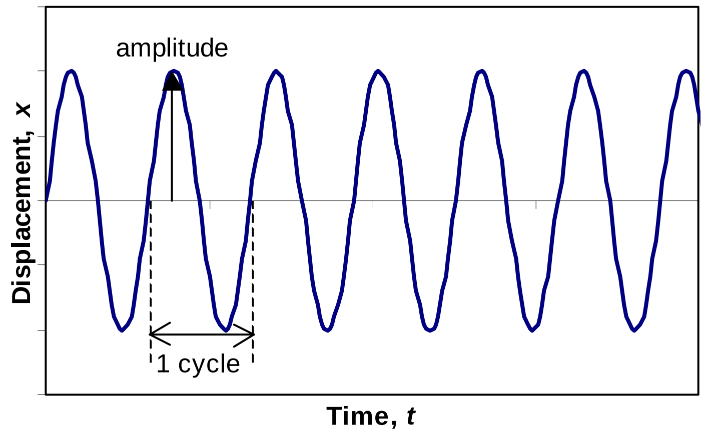


Figure 1.4. The displacement of a single DOF spring-mass system as a function of time.

From it we can define:

The **amplitude** of vibration, A (or X_0, in this example) – which is the maximum displacement achieved.

One cycle is completed when the weight, after starting at one position, returns to the same position and is moving in the same direction.

The **natural frequency** of vibration is $\omega = \sqrt{\frac{k}{m}}$ in radians per second.

The **time period** is the time taken for one cycle $T = \frac{2\pi}{\omega}$ (1.5)

The **frequency** of vibration, in cycles per second, $f = \frac{1}{T} = \frac{\omega}{2\pi}$ (1.6)

From the above example, we can see that the amplitude of vibration depends on the initial displacement given to the weight. In general, the amplitude depends on the energy input to a system initially.

The natural frequency depends only on the stiffness and the mass. If we were to use a spring of greater stiffness, the natural frequency would increase, while if we were to add more mass to the spring, the natural frequency would decrease.

There are two ways of describing the frequency of a vibrating system:

(i) In radians per second (ω), and this is sometimes called the "circular" frequency

(ii) In cycles per second (also called Herz (Hz), pronounced "hurts"), (f).

The two are related by equation (1.6). Clearly as the frequency gets bigger, the time taken for one cycle (T) gets smaller. The time period is measured in seconds.

For a system that undergoes rotary motion, there is an analogous solution. Suppose that a disc of moment of inertia (second moment of mass) I_o about its centre, O, is suspended from a fixed point by a rod of torsional stiffness K (Nm rad^{-1}), as shown in Figure 1.5(a).

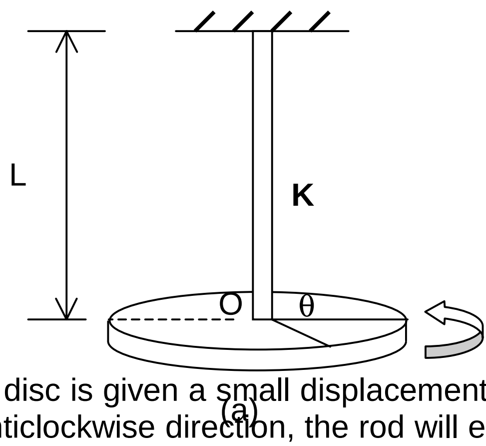


Figure 1.5. (a) Disc suspended by an elastic rod

If the disc is given a small displacement, θ, in an anticlockwise direction, the rod will exert a torque on the disc, $T = K\theta$, as shown in the free body diagram in Figure 1.5(b)

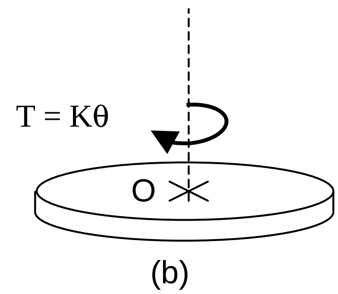


Figure 1.5 (b) FBD of disc when given a small anticlockwise displacement, θ.

From Newton' second law,
$\Sigma M = T = -K\theta$

$$\therefore\ -K\theta = I_o\ddot{\theta}$$

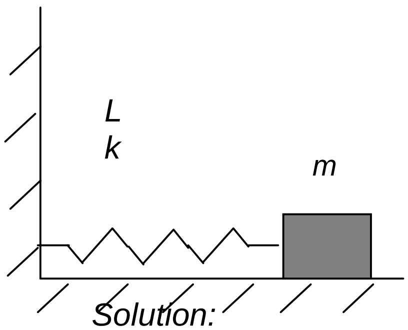
$$\therefore\ I_o\ddot{\theta} + K\theta = 0 \qquad (1.5)$$

Equation (1.5) has exactly the same form as equation (1.2), but I_o replaces m, θ replaces x, and K replaces k. The two systems are analogous. The natural frequency of the rotary system is $\omega = \sqrt{\dfrac{K}{I_o}}$

> **Summary:**
>
> For a simple spring-mass system, the equation of motion of the mass is $m\ddot{x} + kx = 0$
>
> The displacement is $x = A\sin(\omega t + \varepsilon)$
>
> where A is the amplitude, and ε is a constant; A and ε depend on the initial conditions
>
> and ω is the natural frequency, given by $\omega^2 = \dfrac{k}{m}$

Examples

Ex 1. A mass of 2 kg lies on a smooth horizontal table, and is attached by a spring to a fixed point, as shown in the figure below. The spring has a stiffness of 100 Nm^{-1}.



(a) If the mass is given a small displacement to the right, and is then released, neglecting all frictional effects, what will be the natural frequency of the ensuing motion (i) in rad s^{-1} (ii) in Hz?

(b) If, instead, the mass is struck lightly, so that it is given an initial velocity of 2 ms^{-1}. What will be the equation for the ensuing motion.

(c) If a second mass of 0.5 kg is fixed on top of the first, how will the natural frequency be changed? What will be the new natural frequency, in Hz?

Solution:

(a) (i) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{2}} = 7.07$ rad s^{-1}

(ii) $f = \frac{\omega}{2\pi} = \frac{7.071}{2\pi} = 1.1$ Hz

(b) In general, the equation of motion is $x = A\sin(\omega t + \varepsilon)$

Differentiating with respect to time, gives $\dot{x} = \omega A\cos(\omega t + \varepsilon)$

Putting in the initial conditions that at t = 0, x = 0 and $\dot{x} = 2$ ms^{-1}, we get

For x: $0 = A\sin\varepsilon$ $\therefore$ $\sin\varepsilon = 0$ and $\varepsilon = 0$

For $\dot{x}$: $2 = \omega A\cos(\omega t + \varepsilon) = 7.07A$ $\therefore$ $A = \frac{2}{7.07} = 0.283$ m

Therefore, the equation of motion is $x = 0.283\sin(7.07t)$

(c) The natural frequency will reduce: $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{2.5}} = 6.32$ rad s^{-1}

Ex 2. Measurements with a torsion pendulum, using a bronze wire of unknown torsional stiffness, showed that the natural frequency of a solid disc of mass 1.5 kg and diameter 100 mm was 11 Hz. The disc was suspended by the wire at its centre, and vibrated in a manner similar to the system shown in Figure 1.5. Calculate the stiffness of the wire.

Solution: From equation (1.5), $\omega = 2\pi f = \sqrt{\frac{K}{I_o}}$

$$\therefore 4\pi^2 f^2 = \frac{K}{I_o}$$

$$\therefore K = 4\pi^2 f^2 I_o$$

For a circular disc of mass m and radius r, $I_o = \frac{1}{2}mr^2$ (See, for example, the table inside the back cover of "Engineering Mechanics", by Hibbeler.)

$$\therefore K = 4\pi^2 (11)^2 \left(\frac{1}{2}\right)(1.5)\left(\frac{0.1}{2}\right)^2 = 8.96 \text{ Nm}^{-1}$$