MECH 226 - Free Vibration - Section 1B

3. Free damped vibration

In theory, the single degree-of-freedom spring-mass system described above, once set into motion, would continue to move up and down for ever. In practice all systems are damped, which means that energy is dissipated, and the amplitude of the motion gradually gets smaller and smaller until it stops altogether. Damping can be introduced from various sources, and is hard to model accurately. One model, known as viscous damping, is that of a force that is proportional to the velocity of the mass, and which opposes its motion. This is represented by a dashpot, which is given the symbol shown in Fig. 1.6.

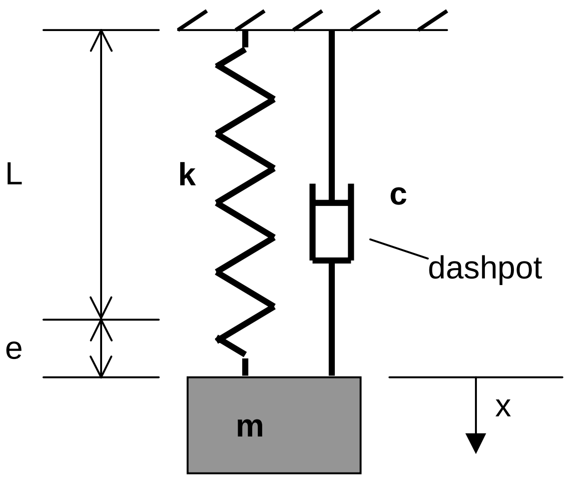


Figure 1.6. A damped single degree-of-freedom system

The constant, *c*, is called the *damping coefficient*, and is the constant by which the velocity is multiplied to give the damping force. Thus, if the displacement from the equilibrium position is *x*, and the velocity $\dot{x}$, the free body diagram of the weight is as shown in Figure 1.7.

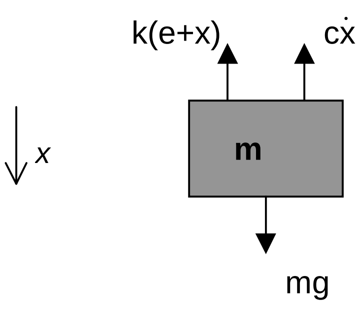


Figure 1.7. FBD of weight suspended by a spring, with damping

Applying Newton's second law,

$\Sigma F_x = mg - k(e + x) - c\dot{x} = ma$

The equation of motion is, therefore,

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\therefore \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

We can write this as:

$$\therefore \ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0 \qquad (1.6)$$

where $$\zeta = \frac{c}{2\sqrt{km}} \qquad (1.7)$$

and, as before, $\omega^2 = \dfrac{k}{m}$

Equation (1.6) is another second order differential equation, which can be shown to have the general solution:

$$x = e^{-\zeta\omega t}\left(C_1 e^{\omega t\sqrt{\zeta^2-1}} + C_2 e^{-\omega t\sqrt{\zeta^2-1}}\right) \qquad (1.8)$$

We need to consider the three following cases:

(1) $\zeta > 1$

In this case the square root of (ζ^2 – 1) is a real number and equation (1.8) indicates that the displacement will vary exponentially with time. There will be no oscillatory motion – no vibration. This is known as an *overdamped* system.

(2) $\zeta = 1$

This is known as a *critically damped* system, and represents the case where the system is *just* non-oscillatory. The displacement gradually returns to its initial value, with no vibration. It can be shown that the solution in this case is:

$$x = (A + Bt)e^{-\omega t} \text{ where A and B are constants} \qquad (1.9)$$

(3) $\zeta < 1$

In this case the square root of (ζ^2 – 1) is an imaginary number and can be written as $\pm i\sqrt{1-\zeta^2}$. Since $e^{ix} = \cos x + i\sin x$, it is possible to write the solution in the form: $x = Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)$ (1.10)

where, as before, the constants A and ε depend on the initial conditions, and

$$\omega_d = \omega\sqrt{1-\zeta^2} \qquad (1.11)$$

ω_d is the damped natural frequency.

Such a system is called an *underdamped* system and will vibrate when released.

To summarise, if $\zeta > 1$, the system is heavily damped, and no vibration occurs. If $\zeta = 1$, the system is critically damped, and the damping is only just sufficient to prevent vibration. If $\zeta < 1$, as it is in the majority of cases, there is insufficient damping in the system to prevent vibration and the motion is oscillatory.

Let us return to the system in Figure 1.6, and, as before, assume that the weight is given an initial displacement, X_0, and then released from rest. Let X_0 = 100 mm.

Therefore, the initial conditions are that at t = 0, x = 100 mm, and $\dot{x} = 0$.

<u>Case (1): Overdamped system with $\zeta = 2$</u>

The displacement is given by $x = e^{-\zeta\omega t}\left(C_1 e^{\omega t\sqrt{\zeta^2-1}} + C_2 e^{-\omega t\sqrt{\zeta^2-1}}\right)$ (equ (1.8)).

Differentiating with respect to time can be shown to give (check it for yourself):

$$\dot{x} = e^{-\omega t}\left[(\lambda - \zeta\omega)C_1 e^{\lambda t} + (-\lambda - \zeta\omega)C_2 e^{-\lambda t}\right] \qquad \text{where } \lambda = \omega\sqrt{\zeta^2 - 1}$$

Therefore, putting t = 0: $x = C_1 + C_2 = 100$ mm (i)

$$\dot{x} = \left[(\lambda - \zeta\omega)C_1 + (-\lambda - \zeta\omega)C_2\right] = 0 \qquad \text{(ii)}$$

For simplicity, suppose that ω = 1 rad s^{-1}, then with ζ = 2, we obtain λ = 1.732 rad s^{-1} and condition (ii) becomes:

$$-0.268C_1 - 3.732C_2 = 0$$

Solving these two simultaneous equations yields:

$$C_1 = -7.74 \text{ mm and } C_2 = 107.74 \text{ mm.}$$

The displacement is shown as a function of time in Figure 1.8(a). The damping force is such as to cause an "undershoot" after which the displacement gradually returns to zero.

Case (2): Critically damped system, ζ = 1

The displacement is given by equation (1.9): $x = (A + Bt)e^{-\omega t}$

Differentiating with respect to time: $\dot{x} = e^{-\omega t}(B - A\omega - B\omega t)$

At t = 0, x = A = 100 mm

$\dot{x} = B - A\omega = 0$

If, as before, ω = 1 rad s^{-1}, then these conditions yield A = 100 mm, and B = 100 mms^{-1}. The motion is shown in Figure 1.8(b). There is no oscillation, and the displacement gradually approaches zero.

Case (3): Underdamped system with ζ = 0.05

The displacement is given by equation (1.10): $x = Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)$

Differentiating with respect to time gives:

$$\dot{x} = Ae^{-\zeta\omega t}\omega_d \cos(\omega_d t + \varepsilon) - \zeta\omega Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)$$

At t = 0, $x = A\sin\varepsilon = X_0 = 100$ (i)

$\dot{x} = A\omega_d \cos\varepsilon - \zeta\omega A\sin\varepsilon = 0$ (ii)

From condition (ii), we find that $\tan\varepsilon = \dfrac{\omega_d}{\zeta\omega}$

Whence (see Figure 1.9) $\sin\varepsilon = \dfrac{\omega_d}{\sqrt{\omega_d^2 + \zeta^2\omega^2}}$

But $\omega_d^2 = \omega^2(1 - \zeta^2) = \omega^2 - \omega^2\zeta^2$

$\therefore\ \omega_d^2 + \omega^2\zeta^2 = \omega^2$

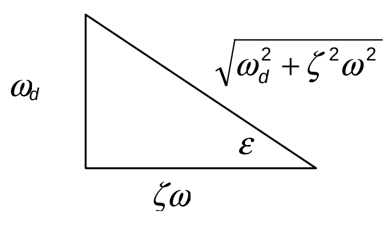


Figure 1.9

$\therefore \sin\varepsilon = \dfrac{\omega_d}{\sqrt{\omega^2}} = \sqrt{1 - \zeta^2} = \sqrt{1 - 0.05^2} = 0.999$ and A = 100/0.999 = 100.1 mm

$\varepsilon = \sin^{-1}(0.999) = 1.526$ rad

The displacement x, is therefore given by: $x = 100.1e^{-0.05t}\sin(0.999t + 1.526)$

This is shown as a function of time in Figure 1.8(c). The system oscillates with an amplitude that decays exponentially with time. The exponential constant ($\zeta\omega$) is sometimes called the damping factor.

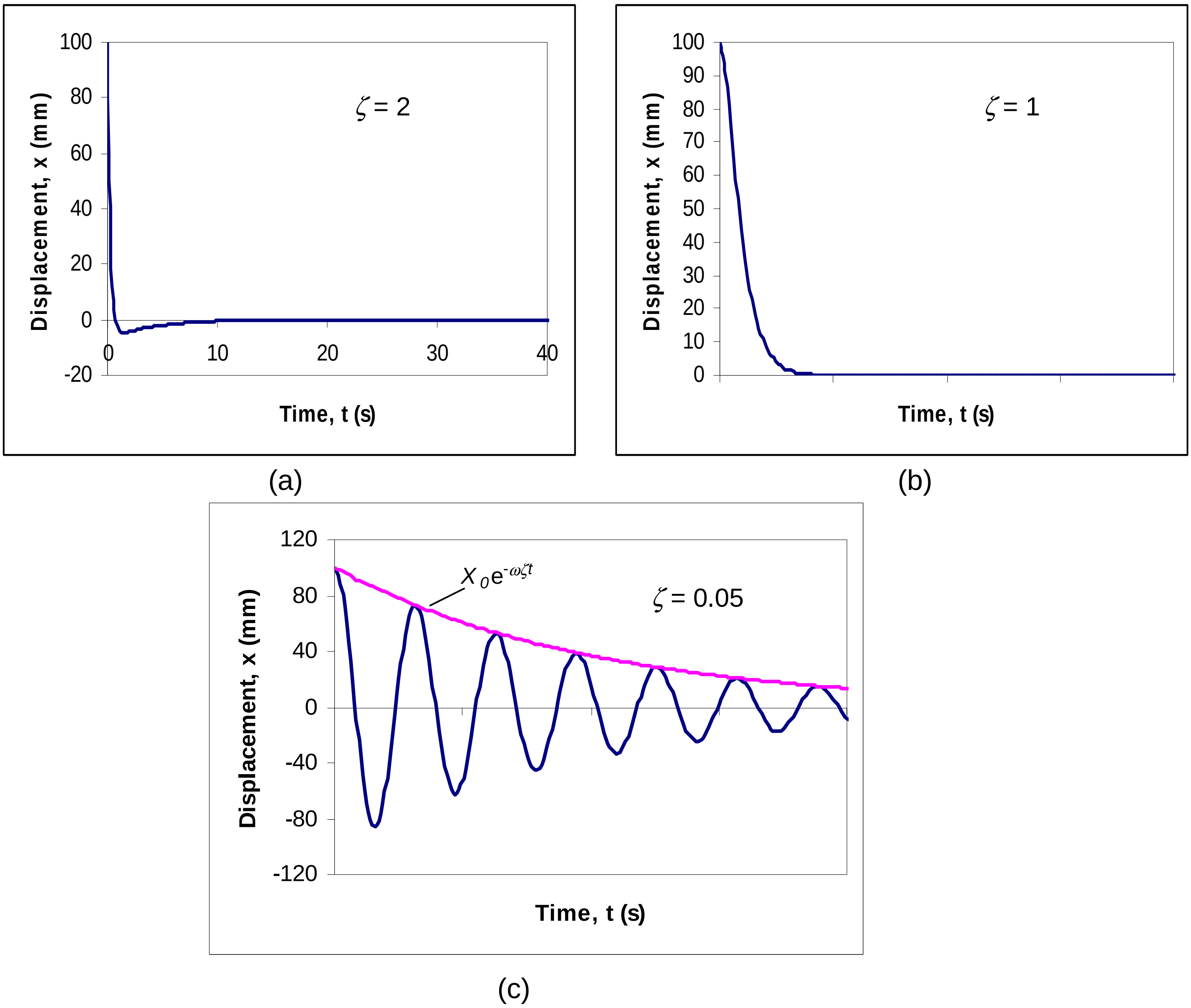
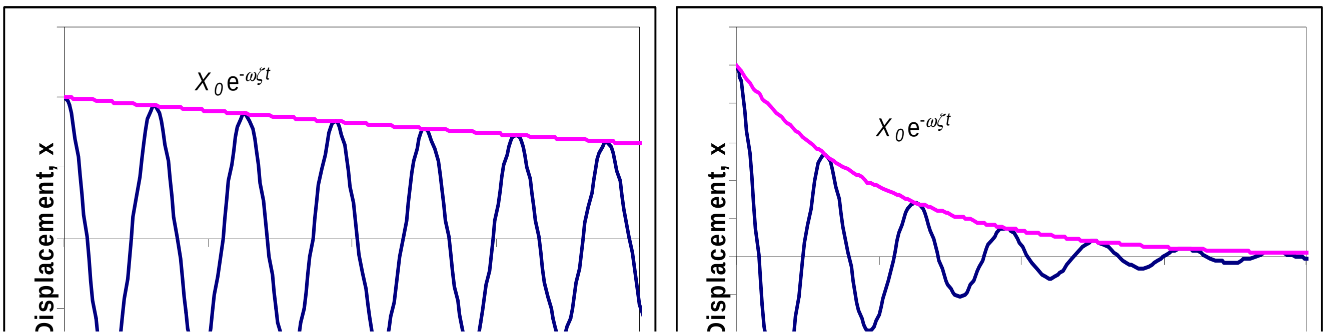


Figure 1.8 The displacement of a freely vibrating system as a function of time for different amounts of viscous damping.

The effect of changing the amount of damping in an underdamped system, by changing the value of ζ is illustrated in Figure 1.9 below. The greater the damping the more quickly the vibrations die away.



The rate at which the amplitude decays gives us another measurement of the damping in a system, known as the *logarithmic decrement*, δ. This is defined as the natural logarithm of the ratio of any two successive amplitudes. In general, we have vibration at a frequency $\omega_d = \omega\sqrt{1-\zeta^2}$, so the time for one cycle is $T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega\sqrt{1-\zeta^2}}$. The amplitude of the vibration is $Ae^{-\zeta\omega t}$

At a time t_1, therefore, the amplitude will be $A_1 = Ae^{-\zeta\omega t_1}$ and after one more cycle, at a time $\left(t_1 + \frac{2\pi}{\omega\sqrt{1-\zeta^2}}\right)$ it will be $A_2 = Ae^{-\zeta\omega\left(t_1 + \frac{2\pi}{\omega\sqrt{1-\zeta^2}}\right)}$

The ratio is, therefore, $\frac{A_1}{A_2} = \frac{Ae^{-\zeta\omega t_1}}{Ae^{-\zeta\omega\left(t_1 + \frac{2\pi}{\omega\sqrt{1-\zeta^2}}\right)}} = e^{\frac{\zeta\omega 2\pi}{\omega\sqrt{1-\zeta^2}}} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$

Taking the natural logarithm of this ratio, the logarithmic decrement,

$$\delta = \ln\left(\frac{A_1}{A_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad (1.12)$$

Rearranging to make ζ the subject: $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$ (1.13)

Summary

For a simple spring-mass system, with viscous damping, the equation of motion of the mass is $m\ddot{x} + c\dot{x} + kx = 0$

The displacement is $x = Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)$ for $\zeta < 1$

where $\omega^2 = \frac{k}{m}$, $\zeta = \frac{c}{2\sqrt{km}}$, and, therefore, $\zeta\omega = \frac{c}{2m}$.

The frequency of the damped vibration is $\omega_d = \omega\sqrt{1-\zeta^2}$ which is slightly lower than that of the undamped natural vibration, ω.

The amplitude decays exponentially at a rate determined by $\zeta\omega$.

The logarithmic decrement, δ is the natural logarithm of the ratio of the amplitudes of successive cycles: $\delta = \ln\left(\frac{A_1}{A_2}\right)$

Thus 3 parameters give a measure of the damping in the system:

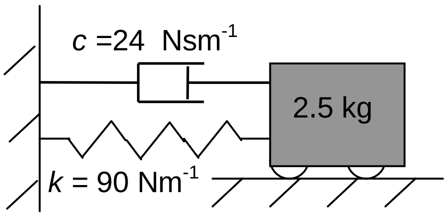
c	the damping coefficient ($N(ms^{-1})^{-1}$ or Nsm^{-1})
ζ	the damping ratio (dimensionless)
δ	the logarithmic decrement (dimensionless)

Critical damping occurs when $\zeta = 1$, while $\zeta > 1$ produces an overdamped system.

This may all seem very mathematical, but in practice, the concepts are quite simple. A system which possesses mass and elasticity can vibrate. Damping in the system causes the amplitude of the vibrations to decrease with time. The more damping in the system, the more rapidly the amplitude dies away. If there is excessive damping no vibration will take place. The damping in the system can be determined by measuring the rate at which the amplitude decays, and thereby calculating the logarithmic decrement, δ. This is related to the damping ratio, ζ (equation (1.12)), which in turn is a function of the mass, the stiffness, and the damping coefficient, *c*. (equation (1.7)) which are used to model the system.

Examples

Ex 3.



Determine the value of (a) the natural frequency and (b) the damping ratio for the simple spring-mass-dashpot system shown.

Solution:

(a) From equation (1.4): $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{90}{2.5}} = 6 \text{ rad s}^{-1}$

(b) From equation (1.7): $\zeta = \frac{c}{2\sqrt{km}} = \frac{24}{2\sqrt{90 * 2.5}} = 0.8$

So the mass will oscillate if given an initial displacement and released, but the motion will be damped out quite quickly, since ζ is not much less than 1.

Ex 4. The 2.5 kg mass in Example 3 is released from rest at a distance 10 mm to the right of the equilibrium position. Determine the displacement as a function of time, and estimate how long it will take for the amplitude of the motion to be reduced to less than 1 mm.

Solution:

Since ζ < 1, the system is underdamped. The solution to the equation of motion is therefore given by equation (1.10): $x = Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)$ where the damped frequency $\omega_d = \omega\sqrt{1-\zeta^2}$ (equation (1.11))

The initial conditions are that at t = 0, x = 10 mm = 0.01 m and $\dot{x} = 0$

From Example 3, the natural frequency ω = 6 rad s^{-1}

$\therefore \quad \omega_d = 6 * \sqrt{1-0.8^2} = 3.6$ rad s^{-1}

Differentiating with respect to time, we find that

$\dot{x} = A\left[\omega_d e^{-\zeta\omega t}\cos(\omega_d t + \varepsilon) - \zeta\omega e^{-\zeta\omega t}\sin(\omega_d t + \varepsilon)\right]$

So, putting in the initial conditions:

For x: $0.01 = A\sin\varepsilon$ (i)

For $\dot{x}$: $0 = A[\omega_d\cos\varepsilon - \zeta\omega\sin\varepsilon] = A(3.6\cos\varepsilon - (0.8*6)\sin\varepsilon)$ (ii)

From (ii), $3.6\cos\varepsilon = 4.8\sin\varepsilon$ because A does not equal 0.

$$\therefore \tan\varepsilon = \frac{3.6}{4.8} = 0.75 \qquad \therefore \varepsilon = 36.9^\circ = 0.644 \text{ rad}$$

Putting this value for ε in (i): $A = \dfrac{0.01}{\sin 36.9^o} = 0.017$ m

The displacement is, therefore,
$x = Ae^{-\zeta\omega t}\sin(\omega_d t + \varepsilon) = 0.017e^{-4.8t}\sin(3.6t + 0.644)$

The amplitude of the response is given by the exponential term $Ae^{-\zeta\omega t}$ = $0.017e^{-4.8t}$

So, if the amplitude is 1 mm = 0.001 m, then 0.001 = $0.017e^{-4.8t}$

$$\therefore e^{-4.8t} = \frac{0.001}{0.017} = 0.0588$$

Taking the natural log of both sides of this equation,
$-4.8t = \ln(0.0588) = -2.836 \qquad \therefore t = 0.59$ s

The amplitude will become less than 1 mm, therefore, after 0.59 seconds only.

Ex 5. A single degree of freedom system, having a mass of 2.4 kg, is set into motion with viscous damping, and allowed to oscillate freely. The frequency of the oscillation is found to be 15 Hz and measurement of the amplitude of oscillation shows two successive amplitudes to be 5.5 mm and 5.1 mm. Determine the viscous damping coefficient, *c*.

The logarithmic decrement, from equ (1.12) is $\delta = \ln\left(\frac{A_1}{A_2}\right) = \ln\left(\frac{5.5}{5.1}\right) = 0.0755$

From equ (1.13), the damping ratio is $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx \frac{\delta}{2\pi}$ for small δ

$$\therefore \quad \zeta = \frac{0.0755}{2\pi} = 0.012$$

The damped natural frequency is given as 15 Hz.

$$\therefore \quad \omega_d = 2\pi f_d = 2\pi * 15 = 94.25 \text{ rad s}^{-1}$$

But from equ(1.11), $\omega_d = \omega\sqrt{1-\zeta^2} \approx \omega$ for small ζ.

$$\therefore \quad \omega = 94.25 \text{ rad s}^{-1}$$

To find *k*, use the relationship in equation (1.4) $\omega = \sqrt{\frac{k}{m}}$ from which

$$k = m\omega^2 = 2.4 * 94.25^2 = 21319 \text{ Nm}^{-1}$$

By definition, $\zeta = \frac{c}{2\sqrt{km}}$

$$\therefore \quad c = 2\zeta\sqrt{km} = 2 * 0.012 * \sqrt{21319 * 2.4} = 5.43 \text{ Nsm}^{-1}$$