MECH226 VIBRATION

Section 2. Forced vibration of a single degree-of-freedom system

In Section 1, we saw that when a system is given an initial input of energy, either in the form of an initial displacement or an initial velocity, and then released it will, under the right conditions, vibrate freely. If there is damping in the system, then the oscillations die away. If a system is given a continuous input of energy in the form of a continuously applied force or a continuously applied displacement, then the consequent vibration is called forced vibration. The energy input can overcome that dissipated by damping mechanisms and the oscillations are sustained.

We will consider two types of forced vibration. The first is where the ground to which the system is attached is itself undergoing a periodic displacement, such as the vibration of a building in an earthquake. The second is where a periodic force is applied to the mass, or object performing the motion; an example might be the forces exerted on the body of a car by the forces produced in the engine. The simplest form of periodic force or displacement is sinusoidal, so we will begin by considering forced vibration due to sinusoidal motion of the ground.

In all real systems, energy will be dissipated, i.e. the system will be damped, but often the damping is very small. So let us first analyse systems in which there is no damping.

(a) Undamped forced vibration – ground excitation

The simplest single degree-of-freedom (DOF) system is the spring-mass system shown in Figure 2.1. Again, the extension of the spring when the mass hangs in equilibrium is e, so that $ke = mg$. Suppose that the support moves with a sinusoidal displacement, $y = Y_0 \sin\Omega t$. Y_0 is the amplitude of the excitation, and Ω is the excitation frequency. The mass will respond with a displacement x from its equilibrium position. This means that the resulting extension of the spring at any time is (e+x-y). The free body diagram (FBD) of the weight is shown in Figure 2.2.

Figure 2.1. Simple spring-mass system with ground excitation

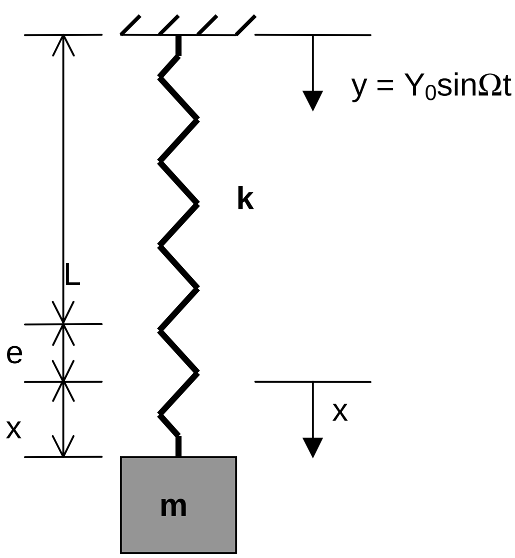
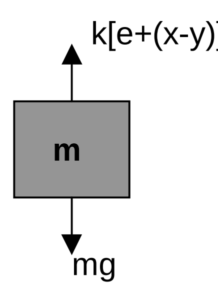


Figure 2.2. FBD of the weight



The equation of motion of the weight is, therefore:

$$\sum F_x = mg - k[e + (x - y)] = m\ddot{x}$$

$$\therefore \quad ky = m\ddot{x} + kx$$

Dividing all through by m, and rewriting: $\ddot{x} + \frac{k}{m}x = \frac{k}{m}Y_0 \sin \Omega t$

$$\therefore \quad \ddot{x} + \omega^2 x = \omega^2 Y_0 \sin \Omega t \qquad (2.1)$$

where, as before, ω is the natural frequency, $\sqrt{\frac{k}{m}}$.

This is a standard second-order differential equation, and the solution is made up of two parts, the complementary solution and the particular solution. The complementary solution is obtained by setting the right-hand side of the equation equal to 0. This is exactly the same equation as for free vibration, and therefore has the solution $A\sin(\omega t+\varepsilon)$, where A and ε depend on the initial conditions. Although the system is modelled as undamped, all real systems do possess damping, so that the free vibration will, eventually die out. Of greater interest, therefore, is the particular solution, which represents the steady state response, i.e. the response that is maintained for as long as the excitation.

Try a particular solution of the form $x = X_0\sin\Omega t$ and see whether this satisfies the equation of motion:

If $x = X_0\sin\Omega t$, then differentiating with respect to time will give:

$$\dot{x} = \Omega X_0 \cos \Omega t \text{ and } \ddot{x} = -\Omega^2 X_0 \sin \Omega t$$

Substituting into equation (2.1): $-\Omega^2 X_0 \sin \Omega t + \omega^2 X_0 \sin \Omega t = \omega^2 Y_0 \sin \Omega t$

Dividing all through by $\sin\Omega t$, since, generally, this does not equal 0:

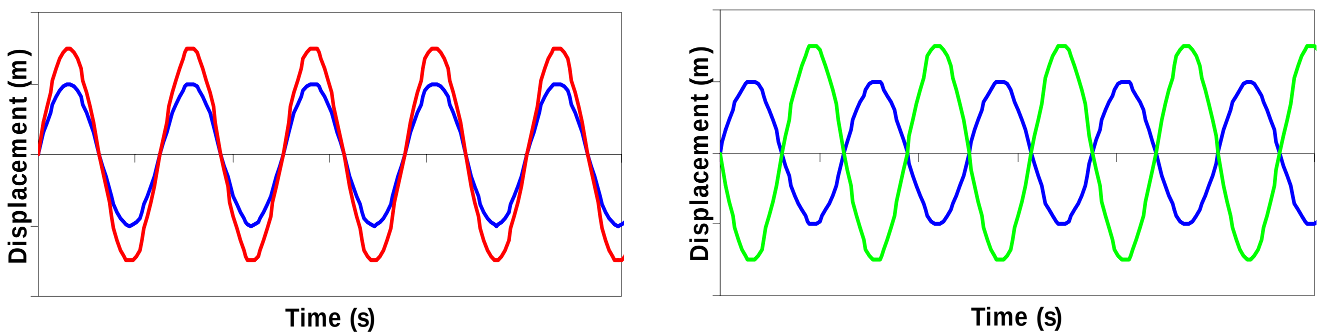
$$-\Omega^2 X_0 + \omega^2 X_0 = \omega^2 Y_0$$

$$\therefore \quad X_0(\omega^2 - \Omega^2) = \omega^2 Y_0$$

$$\therefore \quad X_0 = \frac{\omega^2}{\omega^2 - \Omega^2} Y_0 = \frac{Y_0}{1 - r^2} \qquad (2.2)$$

where $r = \frac{\Omega}{\omega}$, the ratio of the excitation (or driving) frequency to the natural frequency (also known as the frequency ratio).

Thus, $x = X_0\sin\Omega t$ is a solution, provided X_0 is given by equation (2.2). So equation (2.2) tells us how the amplitude of the response varies with the driving frequency Ω. If $\Omega > \omega$, $r > 1$, and equation (2.2) predicts that the amplitude of the response will be negative. This merely means that it will be in the opposite direction to the excitation displacement. Another way to express this is to introduce the idea of phase. If two sinusoidal displacements are in phase, they move together, while if they are exactly out of phase, one will reach its maximum value at the same time as the other reaches its minimum value, as illustrated in Figure 2.3.



(a) (b)

Figure 2.3. (a) Two sine waves exactly in phase $\phi = 0$ (b) two sine waves exactly out of phase $\phi = \pi$

Since $A\sin(\Omega t-\pi) = -A\sin(\Omega t)$, the negative amplitude can be described by a phase angle, ϕ, where $\phi = \pi$.

So we can write the solution as $x = X_0\sin(\Omega t-\phi)$

where X_0 is always positive and ϕ is the phase angle, which is 0 for $\Omega < \omega$ and π for $\Omega > \omega$. The response is said to lag the excitation by ϕ.

What about when $\Omega = \omega$? Well, then, from equation (2.2), $X_0 = \infty$, and the amplitude becomes infinitely large. This phenomenon is called *resonance*, and the natural frequency is also the resonant frequency. In other words, in an undamped system, if we try to drive it at its natural frequency the system resonates and the amplitude of the response becomes *very* large. To see how the amplitude of the response varies with the driving frequency, we can plot X_0 against r (Figure 2.4).

(a)

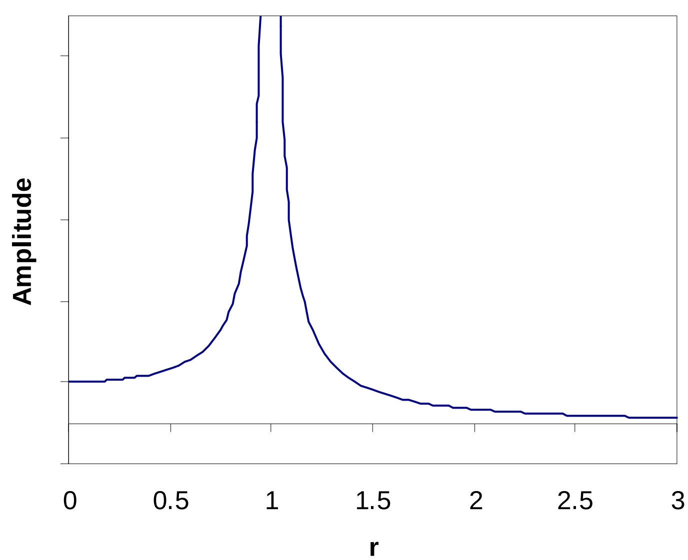
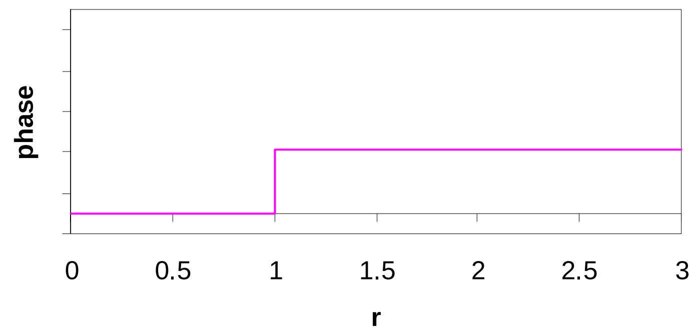


Figure 2.4. Response amplitude and phase of a single DOF system (undamped) as a function the frequency ratio, *r*.

(b)



The magnitude of the , *T*.

$$\therefore \quad T = \left|\frac{X_0}{Y_0}\right| = \left|\frac{1}{1-r^2}\right|$$

T represents the multiple of the driving displacement amplitude (the input) that is transmitted through the spring to the produce the response amplitude of the mass (the output). It is, therefore, a ratio of output to input.

(b) Undamped forced vibration – applied force

We now consider the case when the vibration is caused by a sinusoidal force $F = F_0 \sin \Omega t$ applied to the mass.

Figure 2.5. Simple spring-mass system with excitation by an applied force

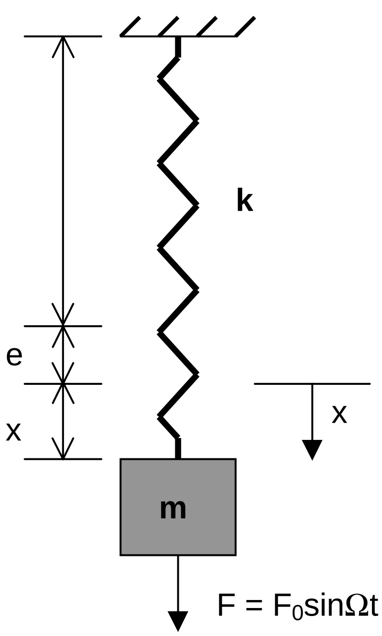
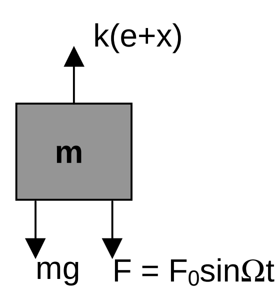


Figure 2.6. FBD of the weight



The system is shown in Figure 2.5, and the free body diagram of the weight in Figure 2.6. Applying Newton's second law in the usual way gives:

$$\sum F_x = mg + F_0 \sin \Omega t - k(e + x) = m\ddot{x}$$

$$\therefore \quad F_0 \sin\Omega t = m\ddot{x} + kx$$

Dividing all through by *m*, and rewriting: $\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin\Omega t$

$$\therefore \quad \ddot{x} + \omega^2 x = \frac{F_0}{m}\sin\Omega t \qquad (2.3)$$

where, as before, ω is the natural frequency, $\sqrt{\frac{k}{m}}$.

If we follow exactly the same reasoning as we did in finding the particular solution to equation (2.1), we find that

$$\therefore \quad X_0 = \frac{\omega^2}{\omega^2 - \Omega^2}\frac{F_0}{k} = \frac{F_0 / k}{1 - r^2} \qquad (2.4)$$

(For details of the analysis, see Appendix 1).

This has exactly the same form as equation (2.2) but F_0/k replaces Y_0. The response amplitude, therefore, varies with *r* in exactly the same way as shown in Figure 2.4. As before, if the frequency of the excitation force is the same as the natural frequency of the system, resonance occurs, and the system vibrates with an infinite amplitude.

Note that F_0/k is the static deflection of the spring when a steady force F_0 is applied. The ratio $\frac{X_0}{F_0 / k}$ is called the magnification, *M*, and represents by how much the static displacement is multiplied when the force is sinusoidal. From equation (2.4), $M = \left|\frac{1}{1 - r^2}\right|$.

Notice that the force transmitted through the spring is *kx* (draw the free body diagram of the support to show this). Suppose the force transmitted is *P*, then $P = kx = kX_0 \sin\Omega t = P_0 \sin\Omega t$ where the amplitude of the transmitted force is $P_0 = kX_0$. The transmissibility is defined as the ratio of the amplitude of the force transmitted to the amplitude of the applied force, so that $T = \frac{P_0}{F_0} = \frac{kX_0}{F_0} = \left|\frac{1}{1 - r^2}\right|$

So the magnification, *M* and the transmissibility *T* have the same value in this case, but there is a subtle difference between them, by their definitions. The transmissibility represents the ratio of the output to the input. So, if the input is excitation by ground *displacement*, then the output is also a *displacement*, and is the amplitude of the response. But if the input is excitation by an applied *force*, then the output is also a *force*, in this case, that transmitted through the spring to the ground.

Summary: Forced vibration with no damping

For a simple spring-mass system, undergoing forced vibration by means of ground excitation, the equation of motion of the mass is

$$m\ddot{x} + kx = kY_0 \sin \Omega t$$

where Ω is the frequency of the excitation (the driving frequency).

The steady state response is $x = X_0 \sin(\Omega t - \phi)$

where the amplitude $X_0 = \left| \frac{Y_0}{1 - r^2} \right|$ and ϕ is the phase lag

$\phi = 0$ if $r < 1$ and $\phi = \pi$ if $r > 1$, r is the frequency ratio $\frac{\Omega}{\omega}$

and ω is the natural frequency, given by $\omega^2 = \frac{k}{m}$

The transmissibility $T = \left| \frac{X_0}{Y_0} \right| = \left| \frac{1}{1 - r^2} \right|$

If the excitation is due to a sinusoidal force $F_0 \sin \Omega t$ applied to the mass, the equation of motion is

$$m\ddot{x} + kx = F_0 \sin \Omega t$$

The steady state response is again $x = X_0 \sin(\Omega t - \phi)$

where the amplitude $X_0 = \left| \frac{F_0 / k}{1 - r^2} \right|$ and ϕ is the phase lag

$\phi = 0$ if $r < 1$ and $\phi = \pi$ if $r > 1$

The magnification $M = \left| \frac{A}{F_0 / k} \right| = \left| \frac{1}{1 - r^2} \right|$

The transmissibility $T = \left| \frac{P_0}{F_0} \right| = \left| \frac{1}{1 - r^2} \right|$

In both cases, when the driving frequency Ω is equal to the natural frequency ω ($r = 1$) the response amplitude becomes infinite. This is known as resonance.

Example 1.

A 750 gram component is mounted via springs to a base which is on springs is expected to vibrate at a frequency of 30 rad s^{-1} with an amplitude of 3 mm. The component is known to have a natural frequency of 22 rad s^{-1} and very little damping. Estimate (a) the spring stiffness (b) the frequency ratio (c) the transmissibility (d) the response amplitude.

Solution:

We have a situation of undamped forced vibration by ground excitation.

Start by writing down all the information you are given:

m = 750 g = 0.75 kg $\qquad \omega$ = 22 rad s^{-1}

Ω = 30 rad s^{-1} $\qquad Y_0$ = 3 mm

(a) $\omega^2 = \frac{k}{m} \quad \therefore \quad k = \omega^2 m = (22)^2 * 0.75 = 363 \text{ Nm}^{-1}$

$\therefore$ the spring stiffness = 363 Nm^{-1}

(b) The frequency ratio: $r = \frac{\Omega}{\omega} = \frac{30}{22} = 1.364$

(c) The transmissibility: $T = \left|\frac{1}{1-r^2}\right| = \left|\frac{1}{1-1.364^2}\right| = 1.162$

(d) The response amplitude: $X_0 = TY_0 = 1.162 * 3 = 3.5$ mm

Example 2

A delicate instrument of mass 500 g is mounted on springs to isolate it from the vibration of the table on which it is placed. The table is found to vibrate with an amplitude of 4 mm at 50 Hz. The instrument is expected to tolerate a maximum amplitude of 0.5 mm. What should be the maximum stiffness of the springs used in the mounting. Assume that there is negligible damping, and that the system behaves as a single degree-of-freedom system.

Solution:

Another case of undamped forced vibration by ground excitation.

Again start by writing down what you know: m = 500 g = 0.5 kg

Y_0 = 4 mm $\qquad \Omega$ = 50 Hz = 2π*50 = 314.2 rad s^{-1} $\qquad X_0 \leq 0.5$ mm.

Let k be the spring stiffness. k determines the natural frequency, so we need to try and find the maximum allowed natural frequency of the system. We know the excitation frequency, therefore we need to know the frequency ratio, *r*.

How to find r? Well, we know both X_0 (its maximum value) and Y_0 and can therefore find the maximum transmissibility, and this is related to *r*. So start by finding *T*, then *r*, then *w* and then *k*.

The transmissibility $T = \frac{X_0}{Y_0} \leq \frac{0.5}{4} \leq 0.125$

So the maximum allowed transmissibility is T = 0.125

T can also be expressed in terms of r: $T = \left|\frac{1}{1-r^2}\right|$ $\therefore \left|1-r^2\right| = \frac{1}{T}$

We know that if r< 1 T is positive, while if r > 1 T will be negative, so we have two situations which will satisfy the conditions:

$1-r^2 = \frac{1}{0.125}$ and $1-r^2 = -\frac{1}{0.125}$

Taking the first, $r^2 = 1 - \frac{1}{0.125} = 1-8 = -7$

But this gives an "imaginary" value for r, so is impossible.

The other condition gives $1-r^2 = -\frac{1}{0.125} = -8$ $\therefore r^2 = 1+8 = 9$

$\therefore r = \sqrt{69} = 3$

If you look at Figure 2.4 you will see that, when r >1, the larger r gets the smaller gets the amplitude. This means that the value we have worked out for r of 3 is the smallest value it can take to keep the response amplitude at the required level. Therefore, we need r ≥ 3.

But $r = \frac{\Omega}{\omega}$ $\therefore \frac{\Omega}{\omega} \geq 3$ $\therefore \Omega \geq 3\omega$ $\therefore \frac{\Omega}{3} \geq \omega$

$\therefore \omega \leq \frac{314.2}{3} = 104.7$ rad s^{-1}.

So the maximum value that ω can have is 104.7 rad s^{-1}.

Now $k = m\omega^2 = 0.5 * 76.9^2 = 5481$ Nm^{-1}. This is the maximum value it can take, since 104.7 rad s^{-1} is the highest permissible value of ω.

Therefore the maximum stiffness of mounting spring stiffness 5.48 kNm^{-1}.

Example 3.

A machine of mass 1000 kg is supported on a vertical flexible mounting, modelled as a single degree-of-freedom system. The mounting has a total stiffness 50 kNm^{-1} but negligible damping. Any horizontal motion of the system should be ignored. In normal operation the machine is subjected to a vertical force $F = F_0 \sin\Omega t$ where the amplitude F_0 is 2500 N. Calculate the response amplitude and the force transmitted to the foundations when the driving frequency is (a) 20 Hz and (b) 2 Hz.

Solution:

This is a case of undamped forced vibration by an applied force.

(a) We know: m = 1000 kg F_0 = 2500 N

Ω = 20 Hz = 2π*20 = 125.7 rad s^{-1} k = 50 kNm^{-1}.

First find the natural frequency $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{50*1000}{1000}} = \sqrt{50} = 7.07$ rad s^{-1}

Then $r = \dfrac{\Omega}{\omega} = \dfrac{125.7}{7.07} = 17.8$

The amplitude of vibration of the engine is

$$X_0 = \left|\frac{F_0/k}{1-r^2}\right| = \left|\frac{2500/(50*1000)}{1-17.8^2}\right| = \left|\frac{0.05}{-315.8}\right| = 1.6*10^{-4} \text{ m} = 0.16 \text{ mm}$$

The amplitude of the force transmitted is

$$|P_0| = \left|\frac{F_0}{1-r^2}\right| = \left|\frac{2500}{1-17.8^2}\right| = \left|\frac{2500}{-315.8}\right| = 7.9 \text{ N}$$

(b) Ω = 2 Hz = 2π*2 = 12.6 rad s^{-1}

$$r = \frac{\Omega}{\omega} = \frac{12.6}{7.07} = 1.8$$

$$X_0 = \left|\frac{F_0/k}{1-r^2}\right| = \left|\frac{2500/(50*1000)}{1-1.8^2}\right| = \left|\frac{0.05}{-2.24}\right| = 0.022 \text{ m} = 22 \text{ mm}$$

$$|P_0| = \left|\frac{F_0}{1-r^2}\right| = \left|\frac{2500}{1-1.8^2}\right| = \left|\frac{2500}{-2.24}\right| = 1116 \text{ N}$$

Reducing the excitation frequency so that r is closer to 1 has increased the amplitude of the response by more than 100 times, and the force transmitted by a similar amount.