Discontinuous Bending Moment Expressions

In the previous examples the bending moment expressions applied to the whole length $0 < x < L$ of the beam. However many bending problems involve transverse loads or couples acting at various points along the beam. Such problems have different bending moment expressions in different regions of the beam.

Consider the simply supported beam shown in Figure 4.

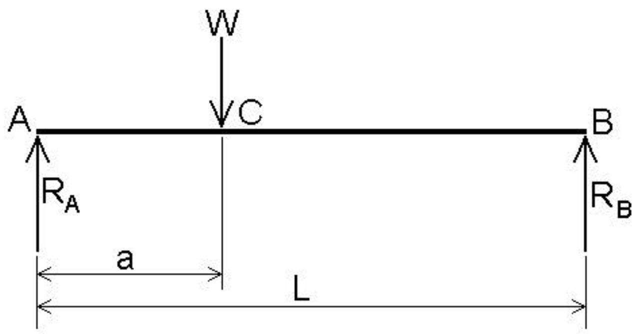

Figure 4.

Let x be the distance from A

Then for the region $0 < x < a$: $M = R_A x$

and for the region $a < x < L$: $M = R_A x - W(x - a)$

Sketching the variation of M with x gives the bending moment diagram shown in Figure 5.

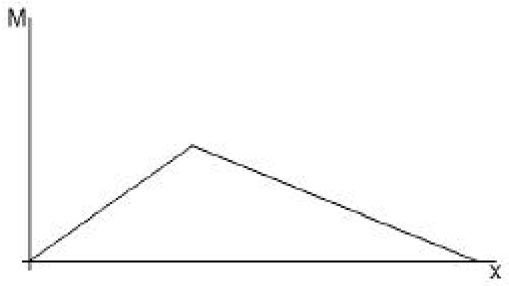

Figure 5.

Mathematically the function has a discontinuity at x = a. It follows that if the deflection is to be found as the solution of:

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

it is necessary to deal with each equation over the appropriate length giving two equations for slope and two for deflection. The solution follows:

For the region 0 < x < a:

$$M = R_A.x$$

Therefore:

$$EI\frac{d^2y}{dx^2} = R_A.x$$

Integrating for slope and deflection:

$$EI\frac{d^2y}{dx^2} = R_A.\frac{x^2}{2} + A_1$$

(i)

$$EIy = R_A.\frac{x^3}{6} + A_1x + B_1$$

(ii)

And for the region a < x < L:

$$M = R_Ax - W(x-a)$$

Therefore:

$$EI\frac{d^2y}{dx^2} = R_Ax - W(x-a)$$

Integrating for slope and deflection:

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - \frac{W(x-a)^2}{2} + A_2$$

(iii)

$$EIy = R_A\frac{x^3}{6} - \frac{W(x-a)^3}{6} + A_2x + B_2$$

(iv)

Assuming that R_A is found by statics there are four unknown constants A_1 B_1 A_2 and B_2 and only two conditions:

- y = 0 at x = 0
- y = 0 at x = L

The extra two equations necessary to solve for the constants are obtained from the observation that equations (i) and (iii) give a common value of slope at C, and equations (ii) and (iv) give a common value of deflection at the same point.

Applying the above conditions to solve for the constants:

At y = 0, x = 0 in equation (ii):

$$0 = B_1$$

(v)

At y = 0, x = L in equation (iv):

$$0 = \frac{R_AL^3}{6} - \frac{W(L-a)^3}{6} + A_2L + B_2$$

(vi)

At x = a:

$$\left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)_3$$

$$R_A\frac{a^2}{2} + A_1 = R_A\frac{a^2}{2} + A_2$$

(vii)

Also:

$$(y)_2 = (y)_4$$

$$R_A \frac{a^3}{6} + A_1 a + B_1 = R_A \frac{a^3}{6} + A_2 a + B_2$$

(viii)

Solving (v), (vi), (vii) and (viii) for A_1 B_1 A_2 and B_2:

From (vii): $A_1 = A_2$ (= A)

Substituting in (viii): $B_1 = B_2$ (= B)

Then from (v): B = 0

Substituting in (vi):

$$A = -\frac{R_A L^2}{6} + \frac{W(L-a)^3}{6L}$$

The expressions for slope and deflection become:

For the range 0 < x < a:

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{R_A L^2}{6} + \frac{W(L-a)^3}{6L}$$

$$EIy = \frac{R_A x^3}{6} + \left(-\frac{R_A L^2}{6} + \frac{W(x-a)^3}{6L} \right) x$$

For the range a < x < L:

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{W(x-a)^2}{2} - \frac{R_A L^2}{6} + \frac{W(L-a)^3}{6L}$$

$$EIy = \frac{R_A x^3}{6} - \frac{W(x-a)^3}{6} + \left(-\frac{R_A L^2}{6} + \frac{W(x-a)^3}{6L} \right) x$$

Inspection of the above solution shows that the expressions for d^2y/dx^2 , dy/dx and y are the same for the two regions except that in the range $a < x < L$ there is an additional term in (x - a).

The solution of problems involving discontinuous functions is simplified by the use of Macaulay's method. This method uses a singularity function which is more generally known as the Unit Step function or the Heaviside function and written:

$$H(x - a) = 0 \quad \text{when } x < a$$

$$H(x - a) = (x - a) \quad \text{when } x .> a$$

In Macaulay's use of this function the H is often omitted and, instead, special singularity or Macaulay brackets are used such that:

$$H(x - a) = [x - a]$$

The use of these brackets allows one equation to describe the variation of bending moment associated with any form of loading.

Macaulay's Method - definition of the general bending moment expression

The use of the singularity function to represent the bending moment is described in the following steps:

i. The position x is taken to any point in the extreme right hand region of the beam;

ii. The moments of the transverse forces and reactions are written using the [ ] notation;

iii. Applied moments need the point of application defined and are written for the example shown in Figure 6 as:

$$M = -R_A x + M_C [x - a]^0$$

ie. for AC:

$$M = -R_A x$$

and for CB

$$M = -R_A x + M_C$$

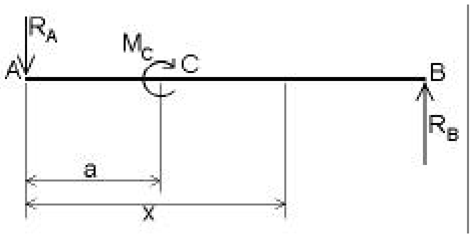

Figure 6.

iv. Distributed loading (Figure 7) requires particular attention - it is dealt with by continuing the loading to the right hand end and adding a negative loading over the extended section as shown in Figure 8.

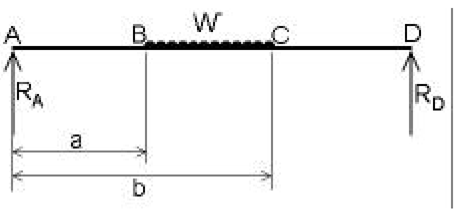

Figure 7.

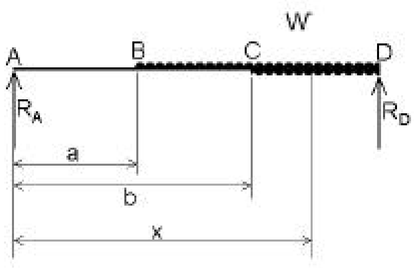

Figure 8.