Slope and Deflection of Beams

Introduction

In the analysis of stresses induced in prismatic beams subjected to bending, the deformation of a cross-sectional plane is considered to allow solution of the statically indeterminate problem. The assumption that transverse planes remain plane during bending led to the introduction of the radius of curvature of the element of beam under consideration.

This led to the result:

$$-\frac{s}{y} = \frac{M}{I} = \frac{E}{R} \tag{1}$$

or:

$$\frac{1}{R} = \frac{M}{EI} \tag{2}$$

Hence if the bending moment M is known at any point the radius of curvature may be determined.

Therefore it is apparent that if R values at every point along the beam are known then the deflected shape can be constructed graphically. In this lecture the analytical method of obtaining an equation for the deflected shape is considered.

Co-ordinate System and Sign Convention

In this analysis the deflected shape of the neutral axis is determined in functional form: y = f(x) where x is the position along the length of the beam and y is the transverse deflection.

The positive co-ordinate directions are taken as x to the right and y upwards as shown in figure 1.

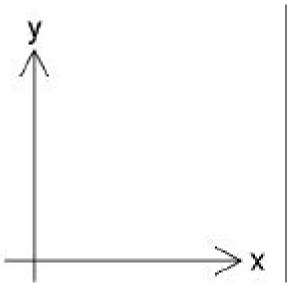


Figure 1

In bending terms this implies that +ve deflection is upwards. Also +ve curvature increases with x giving the shape shown in Figure 2.

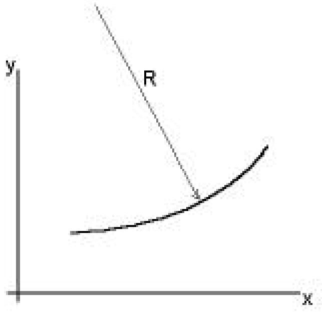


Figure 2.

If the bending equation is used to relate the bending moment to the radius of curvature (2), it is necessary to define a positive bending moment consistent with the positive curvature giving the sense shown in Figure 3.

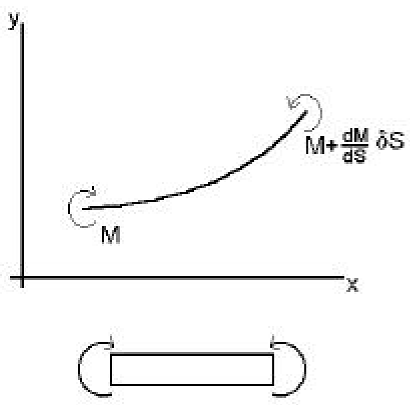


Figure 3.

Moment - Curvature Relationship

Mathematical analysis of curvature in two dimensions gives a relationship for the radius of curvature:

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{3}$$

In many engineering applications the slope induced in a beam due to bending, measured in radians, is significantly less than unity. Hence, (3) may be reasonably approximated to:

$$R = \frac{1}{\frac{d^2y}{dx^2}} \tag{4}$$

and substituting into (2):

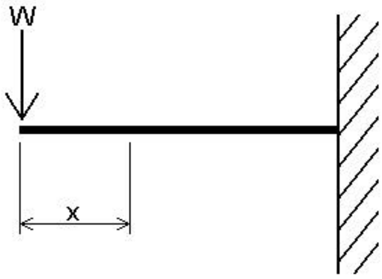
$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{5}$$

This is the equation from which the deflected shape, y = f(x), may be determined.

In cases involving transverse loading, M is a function of x only and the equation may be solved by direct integration.

Example 1

A prismatic bar is clamped at one end and carries a transverse load W at the other. Determine general expressions for the slope and deflection at any point along the bar. Also obtain algebraic values for the slope and deflection at the free end. Take the flexural stiffness as EI and the total length as L.



The bending moment at distance x from the free end is:

$$M_x = -Wx$$

Therefore, from (5):

$$EI\frac{d^2y}{dx^2} = -Wx$$

Integrating for slope (dy/dx):

$$EI\frac{dy}{dx} = -W\frac{x^2}{2} + A$$

Now, when x = L, dy/dx = 0, therefore:

$$A = \frac{WL^2}{2}$$

Integrating again for deflection (y):

$$EIy = \frac{-Wx^3}{6} + \frac{WL^2}{2}x + B$$

Now when x = L, y = 0, therefore:

$$B = \frac{-WL^3}{3}$$

The expressions for slope and deflection become:

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{-Wx^2}{2} + \frac{WL^2}{2} \right)$$

$$y = \frac{1}{EI}\left(\frac{-Wx^3}{6} + \frac{WL^2}{2}x - \frac{WL^3}{3} \right)$$

The algebraic values of slope and deflection are given by the expressions above when x = 0 i.e.:

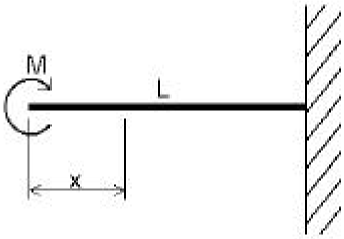
$$\left(\frac{dy}{dx} \right)_{x=0} = \frac{WL^2}{2EI}$$

and

$$(y)_{x=0} = -\frac{WL^3}{3EI}$$

Example 2

Repeat example 1 for when the transverse load W is replaced with a positive bending couple C.



In this case the expression for bending moment is constant:

$$M_x = M$$

Substituting into equation (5):

$$EI\frac{d^2y}{dx^2} = M$$

Integrating for slope:

$$EI\frac{dy}{dx} = Mx + A$$

As in the previous example when x = L, dy/dx = 0, therefore:

$$A = -ML$$

Integrating again for deflection:

$$EIy = \frac{Mx^2}{2} - MLx + B$$

Also, as in the previous example, when x = L, y = 0, therefore:

$$B = \frac{ML^2}{2}$$

The expressions for slope and deflection become:

$$\frac{dy}{dx} = \frac{1}{EI}\left(Mx - ML\right)$$

$$y = \frac{1}{EI}\left(\frac{Mx^2}{2} - MLx + \frac{ML^2}{2}\right)$$

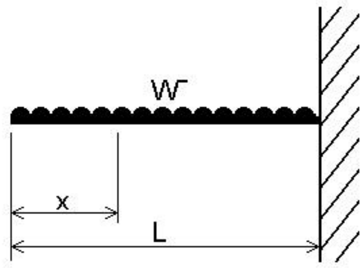
And the algebraic value of slope and deflection at x = 0 are:

$$\left(\frac{dy}{dx}\right)_{x=0} = -\frac{ML}{EI}$$

$$\left(y\right)_{x=0} = -\frac{ML^2}{2EI}$$

Example 3

Repeat example 1 for when the transverse load W is replaced with a uniformly distributed load w over the whole length of the beam.



At distance x from the free end, the expression for bending moment is:

$$M_x = -\frac{Wx^2}{2}$$

Therefore:

$$EI\frac{d^2y}{dx^2} = -\frac{Wx^2}{2}$$

Integrating for slope and deflection as before:

$$EI\frac{dy}{dx} = -\frac{Wx^3}{6} + A$$

$$EIy = -\frac{Wx^4}{24} + Ax + B$$

As in the previous examples, the end conditions are:

- x = L, dy/dx = 0
- x = L, y = 0

From (i):

$$A = \frac{WL^3}{6}$$

From (ii):

$$B = -\frac{WL^4}{8}$$

The expressions for slope and deflection become:

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{Wx^3}{6} + \frac{WL^3}{6}\right)$$

$$y = \frac{1}{EI}\left(-\frac{Wx^4}{24} + \frac{WL^3x}{6} - \frac{WL^4}{8}\right)$$

And the algebraic values at x = 0 become:

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{WL^3}{6EI}$$

$$(y)_{x=0} = -\frac{WL^4}{8EI}$$